

The covariant perturbative string spectrum

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Abstract

We provide generating functions for the perturbative massive string spectrum which are covariant with respect to the $SO(9)$ little group, and which contain all the representation theoretic content of the spectrum. Generating functions for perturbative bosonic, Type II, Heterotic and Type I string theories are presented, and generalizations are discussed.

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1 Introduction

The multiplicities of the perturbative on-shell string states as a function of their mass is known, for instance through the light-cone gauge partition function. For a string theory in $(D - 1, 1)$ Minkowski space, the expression can easily be made $SO(D - 2)$ covariant. Massive string states however are classified on-shell by the little group $SO(D - 1)$. Although it is known that the perturbative string spectrum respects this

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symmetry since it respects Lorentz invariance, no generating functions for the $SO(D-1)$ representation content of string states at all mass levels has been given. It is our purpose in this paper to provide such generating functions.

2 Two tools

In order to write down the generating functions in a compact form, we introduce two tools. Firstly, we discuss characters for irreducible representations characterized by their Dynkin labels. Secondly, we introduce plethystics.

2.1 Characters and Dynkin labels

An irreducible representation of the rotation group $SO(n)$ is characterized by its Dynkin labels, a_1, a_2, \dots, a_r , where r is the rank of $SO(n)$ and the a_i are non-negative integers. The integers give the coefficient of the fundamental weights in the highest weight of the irreducible representation of $SO(n)$. We denote the corresponding character by $[a_1, a_2, \dots, a_r]_n$. For the purpose of this note, the main groups of interest are $SO(8)$ and $SO(9)$. In ten dimensions, these are the little groups for massless and massive particles respectively. Each representation of these groups carries four Dynkin labels. To each fundamental weight we assign a fugacity. Each weight in the irreducible representation gives rise to a term in the character formula which raises the fugacity to the power equal to the coefficient of the fundamental weight in the weight. For $SO(8)$ we denote the four fugacities z_i and for $SO(9)$ we denote them y_i .

Explicit formulae are needed for the characters of the eight-dimensional vector representation with character $[1, 0, 0, 0]_8$, the eight-dimensional spinor representation $[0, 0, 1, 0]_8$ and the eight-dimensional complex conjugate spinor representation $[0, 0, 0, 1]_8$:

$$\begin{aligned} [1, 0, 0, 0]_8 &= z_1 + \frac{z_2}{z_1} + \frac{z_3 z_4}{z_2} + \frac{z_4}{z_3} + \frac{z_3}{z_4} + \frac{z_2}{z_3 z_4} + \frac{z_1}{z_2} + \frac{1}{z_1}, \\ [0, 0, 1, 0]_8 &= z_3 + \frac{z_2}{z_3} + \frac{z_4 z_1}{z_2} + \frac{z_1}{z_4} + \frac{z_4}{z_1} + \frac{z_2}{z_4 z_1} + \frac{z_3}{z_2} + \frac{1}{z_3}, \\ [0, 0, 0, 1]_8 &= z_4 + \frac{z_2}{z_4} + \frac{z_1 z_3}{z_2} + \frac{z_3}{z_1} + \frac{z_1}{z_3} + \frac{z_2}{z_1 z_3} + \frac{z_4}{z_2} + \frac{1}{z_4}. \end{aligned} \quad (2.1)$$

Note that these formulae reflect the $SO(8)$ triality which act by permuting the Dynkin labels a_1, a_3 , and a_4 in addition to the fugacities, z_1, z_3 , and z_4 .

We also give the $SO(9)$ characters corresponding to the nine-dimensional vector representation $[1, 0, 0, 0]_9$, and the sixteen-dimensional spinor representation $[0, 0, 0, 1]_9$:

$$[1, 0, 0, 0]_9 = y_1 + \frac{y_2}{y_1} + \frac{y_3}{y_2} + \frac{y_4^2}{y_3} + \frac{y_3}{y_4^2} + \frac{y_2}{y_3} + \frac{y_1}{y_2} + \frac{1}{y_1} + 1, \quad (2.2)$$

$$\begin{aligned} [0, 0, 0, 1]_9 &= \frac{y_3}{y_4} + \frac{y_2 y_4}{y_3} + \frac{y_4 y_1}{y_2} + \frac{y_1}{y_4} + \frac{y_4}{y_1} + \frac{y_2}{y_4 y_1} + \frac{y_3}{y_2 y_4} + \frac{y_4}{y_3} \\ &+ y_4 + \frac{y_2}{y_4} + \frac{y_1 y_3}{y_4 y_2} + \frac{y_3}{y_1 y_4} + \frac{y_1 y_4}{y_3} + \frac{y_4 y_2}{y_1 y_3} + \frac{y_4}{y_2} + \frac{1}{y_4}. \end{aligned} \quad (2.3)$$

The decomposition of $SO(9)$ representations into irreducible representations of $SO(8)$ can be read off from the characters by relating the $SO(9)$ fugacities to those of $SO(8)$:

$$y_1 = z_1, \quad y_2 = z_2, \quad y_3 = z_3 z_4, \quad y_4 = z_4. \quad (2.4)$$

Thus, using (2.4), the first line in equation (2.3) for $[0, 0, 0, 1]_9$, the character for the spinor representation of $SO(9)$, can be seen to correspond to the $[0, 0, 1, 0]_8$ spinor representation and the second line in equation (2.3) corresponds to the conjugate spinor $[0, 0, 0, 1]_8$. Note that generically, the reconstruction of $SO(9)$ representations from their $SO(8)$ reductions is ambiguous. Nevertheless, for the low-dimensional representations

discussed above, it is clear that the $SO(8)$ weights lift uniquely to $SO(9)$ weights as follows:

$$\begin{aligned}
[1, 0, 0, 0]_8 &= [1, 0, 0, 0]_9 - 1, \\
[0, 0, 1, 0]_8 &= \frac{y_3}{y_4} + \frac{y_2 y_4}{y_3} + \frac{y_4 y_1}{y_2} + \frac{y_1}{y_4} + \frac{y_4}{y_1} + \frac{y_2}{y_4 y_1} + \frac{y_3}{y_2 y_4} + \frac{y_4}{y_3}, \\
[0, 0, 0, 1]_8 &= y_4 + \frac{y_2}{y_4} + \frac{y_1 y_3}{y_4 y_2} + \frac{y_3}{y_1 y_4} + \frac{y_1 y_4}{y_3} + \frac{y_4 y_2}{y_1 y_3} + \frac{y_4}{y_2} + \frac{1}{y_4}.
\end{aligned} \tag{2.5}$$

2.2 Plethystics

Our second tool will be to rewrite various combinatorial expressions in terms of the formalism of plethystics. We collect here the definition of various plethystic functions. Some applications of plethystic functions to problems in string theory and supersymmetric gauge theory can be found in [2] and references thereto.

For a function of m variables $g(t_1, \dots, t_m)$ that vanishes at the origin, $g(0, \dots, 0) = 0$, the plethystic exponential is defined to be

$$PE[g(t_1, \dots, t_m)] = \exp \left(\sum_{k=1}^{\infty} \frac{g(t_1^k, \dots, t_m^k)}{k} \right). \tag{2.6}$$

The fermionic plethystic exponential contains extra minus signs:

$$PE_F[g(t_1, \dots, t_m)] = \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} g(t_1^k, \dots, t_m^k)}{k} \right). \tag{2.7}$$

The inverse of the plethystic exponential is called the plethystic logarithm and is defined for a function of m variables $g(t_1, \dots, t_m)$ that is equal to 1 at the origin, $g(0, \dots, 0) = 1$, as:

$$PL[g(t_1, \dots, t_m)] = \sum_{k=1}^{\infty} \frac{\mu(k) \log g(t_1^k, \dots, t_m^k)}{k}, \tag{2.8}$$

with $\mu(k)$ the Möbius function,

$$\mu(k) = \begin{cases} (-1)^n & k \text{ is a product of } n \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We now have the tools to tackle the generating functions.

3 The covariant perturbative string partition functions

3.1 The chiral ten-dimensional partition function

Let us concentrate on the left-movers of a Type II string, and on the integrand appearing in the partition function (i.e. the integral stripped of both momentum zero-modes and the integration of the modular parameter over the fundamental domain). It is our goal to render the integrand manifestly $SO(9)$ covariant at all massive levels.

The problem

Before introducing fugacities, the partition function takes the form:

$$Z(q) = 16 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^8, \tag{3.1}$$

where q is the fugacity that counts the mass level of the perturbative string spectrum. After combining with the right-movers, the expansion of this function gives the number of physical polarization modes at any given mass. The first few terms in the expansion

$$Z(q) = 16 + 256q + 2304q^2 + 15360q^3 + 84224q^4 + O(q^5), \quad (3.2)$$

give information about the massless and massive spectrum of the open and Type II string. At zeroth order we find the 16 polarization modes of the massless vector multiplet in 9+1 dimensions, which decompose under $SO(8)$ as the vector representation $[1, 0, 0, 0]_8$ and the spinor representation $[0, 0, 0, 1]_8$, corresponding to the 9+1 dimensional gauge field and the 9+1 dimensional gaugino respectively:

$$Z_0 = [1, 0, 0, 0]_8 + [0, 0, 0, 1]_8. \quad (3.3)$$

At the first massive level we find the $SO(9)$ representations which also appear in the massless supergravity multiplet in 10+1 dimensions:

$$Z_Q = [2, 0, 0, 0]_9 + [1, 0, 0, 1]_9 + [0, 0, 1, 0]_9. \quad (3.4)$$

This multiplet corresponds to a multiplet of supercharges and encodes the supersymmetric nature of all massive representations in ten dimensions. Any higher order massive supermultiplet is a tensor product of this multiplet with another representation of $SO(9)$. We can therefore rewrite the partition function in a factorized form

$$Z(q) = 16 + 256qZ_m(q); \quad Z_m(q) = 1 + 9q + 60q^2 + 329q^3 + O(q^4), \quad (3.5)$$

where the first equation serves as the definition of $Z_m(q)$, the partition function for perturbative massive modes, and the second equation is the expansion of $Z_m(q)$ to first few orders. Due to this factorization we can proceed a little further in reconstructing the irreducible representations of $SO(9)$ that appear in the partition function, on the basis of their dimensions only. At the second mass level we find the representation content

$$([2, 0, 0, 0]_9 + [1, 0, 0, 1]_9 + [0, 0, 1, 0]_9) [1, 0, 0, 0]_9, \quad (3.6)$$

and at the third mass level we get

$$([2, 0, 0, 0]_9 + [1, 0, 0, 1]_9 + [0, 0, 1, 0]_9) ([2, 0, 0, 0]_9 + [0, 0, 0, 1]_9). \quad (3.7)$$

This information can also be found in text books. At low levels, the identification of $SO(9)$ representations on the basis of their dimensions only is unique. At higher mass levels this is no longer true.

The solution

To gain further insight, we recall that the perturbative Type II spectrum is made out of a tower of eight bosonic oscillators and eight fermionic oscillators that transform as vectors under $SO(8)$. We can therefore introduce the four fugacities of $SO(8)$ and refine the partition function to include the characters of $SO(8)$ and not just their dimensions. For the massive spectrum, we further wish to extend these characters into $SO(9)$ characters as the massive on-shell spectrum decomposes into irreducible representations of the little group $SO(9)$.

The eight bosonic oscillators transform in the vector representation of $SO(8)$ and carry a level contribution n . The tower of string states is formed by symmetrization of those oscillators. The plethystic exponential precisely keeps track of the symmetrization procedure. The $SO(8)$ covariant bosonic part of the partition function is therefore:

$$Z_B(q; z_1, z_2, z_3, z_4) = PE \left[\frac{q}{1-q} [1, 0, 0, 0]_8 \right]. \quad (3.8)$$

If we set the $SO(8)$ fugacities to one, we recover the infinite denominator in equation (3.1):

$$Z_B(q; 1, 1, 1, 1) = PE \left[\frac{8q}{1-q} \right] = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^8}. \quad (3.9)$$

The (worldsheet) fermionic partition function is slightly more involved. The antisymmetrization can be treated by the fermionic plethystic exponential. To implement GSO, we also wish to keep track of the fermion number of all states, for which we introduce an extra fugacity f :

$$Z_F(q; f; z_1, z_2, z_3, z_4) = PE_F \left[\frac{f}{1-q} [1, 0, 0, 0]_8 \right]. \quad (3.10)$$

Again, after setting the $SO(8)$ fugacities to one we find the infinite numerator in equation (3.1) (supplemented with the fermion number fugacity):

$$Z_F(q; f; 1, 1, 1, 1) = PE_F \left[\frac{8f}{1-q} \right] = \prod_{n=0}^{\infty} (1 + fq^n)^8. \quad (3.11)$$

In the formulas below, the q and z_i dependence is kept implicit. Only the f dependence is mentioned explicitly, namely $Z_F(f)$. We define the GSO projected partition functions in the Neveu-Schwarz

$$Z_{NS} = \frac{1}{2\sqrt{q}} (Z_F(\sqrt{q}) - Z_F(-\sqrt{q})), \quad (3.12)$$

and in the Ramond-sector

$$Z_{\pm} = \frac{1}{2} (Z_F(q) \pm Z_F(-q)). \quad (3.13)$$

With these definitions we can collect the different fermionic partition functions into a single partition function that takes into account the boundary conditions, the GSO projection, and the fugacities associated to the vacua

$$Z_{RNS}(q; z_1, z_2, z_3, z_4) = Z_{NS} + [0, 0, 0, 1]_8 Z_+ + [0, 0, 1, 0]_8 Z_-. \quad (3.14)$$

The open string, or left-moving partition function takes the form

$$Z_{Left}(q; z_1, z_2, z_3, z_4) = Z_B (Z_{NS} + [0, 0, 0, 1]_8 Z_+ + [0, 0, 1, 0]_8 Z_-). \quad (3.15)$$

Equation (3.15) gives the refined partition function of the open string as a function of $SO(8)$ fugacities. It remains to rewrite it in terms of $SO(9)$ characters, once the massless sector is subtracted. This turns out to be a simple task by realizing that there are only 3 $SO(8)$ representations which are involved in computing Z_{Left} . These are the vector and the two spinor representations. Equations (2.5) provide the final ingredient, as it rewrites the characters of these representations in terms of $SO(9)$ fugacities. Thus using (3.15), (3.8), (3.10), (3.12), and (3.13) we arrive at the open string partition function written in terms of $SO(9)$ fugacities. Using the equations for the characters of the spinor representations of $SO(9)$ and $SO(8)$ we can rewrite equation (3.15):

$$Z_{Left}(q; z_1, z_2, z_3, z_4) = \frac{1}{2} ([0, 0, 0, 1]_8 - [0, 0, 1, 0]_8) + Z_B \left(Z_{NS} + \frac{1}{2} [0, 0, 0, 1]_9 Z_F(q) \right). \quad (3.16)$$

We provide explicit expressions for the functions appearing in the formula above:

$$\begin{aligned} Z_B &= PE \left[\frac{q}{1-q} ([1, 0, 0, 0]_9 - 1) \right], \\ Z_{NS} &= \frac{1}{2\sqrt{q}} \left(PE_F \left[\frac{\sqrt{q}}{1-q} ([1, 0, 0, 0]_9 - 1) \right] - PE_F \left[\frac{(-\sqrt{q})}{1-q} ([1, 0, 0, 0]_9 - 1) \right] \right), \\ Z_F(q) &= PE_F \left[\frac{q}{1-q} ([1, 0, 0, 0]_9 - 1) \right]. \end{aligned} \quad (3.17)$$

The first term in the partition function (3.16) has no q dependence and contributes only at the massless level, while the second contribution is manifestly $SO(9)$ covariant. The factor $1/2$ appears to contribute fractional coefficients to irreducible representations. One can show that this is not the case either by explicit expansion as demonstrated in the next section or by the following general logic. Since the arguments of Z_B and $Z_F(q)$ are equal, the contribution to the plethystic exponential gets a factor 2 from odd powers of the argument. This factor 2 in the exponential therefore cancels the $1/2$ order by order. The generating function (3.16) for the full $SO(9)$ covariant content of the perturbative string spectrum is our main result. Our construction makes it very plausible that the generating function has positive integer coefficients for each $SO(9)$ character at each mass level. It is a challenge to prove this explicitly.

The Covariant Partition Function for Massive Modes

With this result at hand, we can also refine the expression for the massive spectrum, with factored massive supermultiplet. We define:

$$Z_m(q; y_1, y_2, y_3, y_4) = \frac{Z_{Left} - Z_0}{qZ_Q}, \quad (3.18)$$

with Z_{Left} and Z_Q given in equations (3.16), and (3.4) and Z_0 equal to the partition function of the massless modes. The expansion of the function Z_m in powers of q and $SO(9)$ characters can be performed explicitly (using a symbolic manipulation program) and gives

$$\begin{aligned} Z_m = & 1 + [1, 0, 0, 0]_9 q + ([2, 0, 0, 0]_9 + [0, 0, 0, 1]_9) q^2 \\ & + ([3, 0, 0, 0]_9 + [1, 0, 0, 1]_9 + [1, 0, 0, 0]_9 + [0, 1, 0, 0]_9) q^3 \\ & + ([4, 0, 0, 0]_9 + [2, 0, 0, 1]_9 + [2, 0, 0, 0]_9 + [1, 1, 0, 0]_9 + [1, 0, 0, 1]_9 + [0, 1, 0, 0]_9 + [0, 0, 1, 0]_9 \\ & + [0, 0, 0, 1]_9 + [0, 0, 0, 0]_9) q^4 \\ & + ([5, 0, 0, 0]_9 + [3, 0, 0, 1]_9 + [3, 0, 0, 0]_9 + [2, 1, 0, 0]_9 + [2, 0, 0, 1]_9 + [2, 0, 0, 0]_9 + 2[1, 1, 0, 0]_9 \\ & + [1, 0, 1, 0]_9 + 2[1, 0, 0, 1]_9 + 2[1, 0, 0, 0]_9 + [0, 1, 0, 1]_9 + [0, 1, 0, 0]_9 + [0, 0, 0, 2]_9 + 2[0, 0, 0, 1]_9) q^5 \\ & + ([6, 0, 0, 0]_9 + [4, 0, 0, 1]_9 + [4, 0, 0, 0]_9 + [3, 1, 0, 0]_9 + [3, 0, 0, 1]_9 + [3, 0, 0, 0]_9 + 2[2, 1, 0, 0]_9 \\ & + [2, 0, 1, 0]_9 + 3[2, 0, 0, 1]_9 + 3[2, 0, 0, 0]_9 + [1, 1, 0, 1]_9 + 2[1, 1, 0, 0]_9 + [1, 0, 1, 0]_9 + [1, 0, 0, 2]_9 \\ & + 4[1, 0, 0, 1]_9 + 2[1, 0, 0, 0]_9 + [0, 2, 0, 0]_9 + 2[0, 1, 0, 1]_9 + 2[0, 1, 0, 0]_9 + 3[0, 0, 1, 0]_9 + [0, 0, 0, 2]_9 \\ & + 2[0, 0, 0, 1]_9 + 2[0, 0, 0, 0]_9) q^6 \\ & + ([7, 0, 0, 0]_9 + [5, 0, 0, 1]_9 + [5, 0, 0, 0]_9 + [4, 1, 0, 0]_9 + [4, 0, 0, 1]_9 + [4, 0, 0, 0]_9 + 2[3, 1, 0, 0]_9 \\ & + [3, 0, 1, 0]_9 + 3[3, 0, 0, 1]_9 + 4[3, 0, 0, 0]_9 + [2, 1, 0, 1]_9 + 3[2, 1, 0, 0]_9 + [2, 0, 1, 0]_9 + [2, 0, 0, 2]_9 \\ & + 5[2, 0, 0, 1]_9 + 3[2, 0, 0, 0]_9 + [1, 2, 0, 0]_9 + 3[1, 1, 0, 1]_9 + 5[1, 1, 0, 0]_9 + 4[1, 0, 1, 0]_9 + 2[1, 0, 0, 2]_9 \\ & + 7[1, 0, 0, 1]_9 + 5[1, 0, 0, 0]_9 + [0, 2, 0, 0]_9 + [0, 1, 1, 0]_9 + 4[0, 1, 0, 1]_9 + 5[0, 1, 0, 0]_9 + [0, 0, 1, 1]_9 \\ & + 2[0, 0, 1, 0]_9 + 3[0, 0, 0, 2]_9 + 4[0, 0, 0, 1]_9 + [0, 0, 0, 0]_9) q^7 \\ & + ([8, 0, 0, 0]_9 + [6, 0, 0, 1]_9 + [6, 0, 0, 0]_9 + [5, 1, 0, 0]_9 + [5, 0, 0, 1]_9 + [5, 0, 0, 0]_9 + 2[4, 1, 0, 0]_9 \\ & + [4, 0, 1, 0]_9 + 3[4, 0, 0, 1]_9 + 4[4, 0, 0, 0]_9 + [3, 1, 0, 1]_9 + 3[3, 1, 0, 0]_9 + [3, 0, 1, 0]_9 + [3, 0, 0, 2]_9 \\ & + 6[3, 0, 0, 1]_9 + 4[3, 0, 0, 0]_9 + [2, 2, 0, 0]_9 + 3[2, 1, 0, 1]_9 + 6[2, 1, 0, 0]_9 + 5[2, 0, 1, 0]_9 + 2[2, 0, 0, 2]_9 \\ & + 10[2, 0, 0, 1]_9 + 9[2, 0, 0, 0]_9 + 2[1, 2, 0, 0]_9 + [1, 1, 1, 0]_9 + 6[1, 1, 0, 1]_9 + 9[1, 1, 0, 0]_9 + [1, 0, 1, 1]_9 \\ & + 6[1, 0, 1, 0]_9 + 6[1, 0, 0, 2]_9 + 12[1, 0, 0, 1]_9 + 5[1, 0, 0, 0]_9 + [0, 2, 0, 1]_9 + 4[0, 2, 0, 0]_9 + [0, 1, 1, 0]_9 \\ & + [0, 1, 0, 2]_9 + 8[0, 1, 0, 1]_9 + 7[0, 1, 0, 0]_9 + 3[0, 0, 1, 1]_9 + 7[0, 0, 1, 0]_9 + 4[0, 0, 0, 2]_9 + 8[0, 0, 0, 1]_9 \\ & + 3[0, 0, 0, 0]_9) q^8 \\ & + ([9, 0, 0, 0]_9 + [7, 0, 0, 1]_9 + [7, 0, 0, 0]_9 + [6, 1, 0, 0]_9 + [6, 0, 0, 1]_9 + [6, 0, 0, 0]_9 + 2[5, 1, 0, 0]_9 \\ & + [5, 0, 1, 0]_9 + 3[5, 0, 0, 1]_9 + 4[5, 0, 0, 0]_9 + [4, 1, 0, 1]_9 + 3[4, 1, 0, 0]_9 + [4, 0, 1, 0]_9 + [4, 0, 0, 2]_9 \\ & + 6[4, 0, 0, 1]_9 + 5[4, 0, 0, 0]_9 + [3, 2, 0, 0]_9 + 3[3, 1, 0, 1]_9 + 7[3, 1, 0, 0]_9 + 5[3, 0, 1, 0]_9 + 2[3, 0, 0, 2]_9 \\ & + 11[3, 0, 0, 1]_9 + 11[3, 0, 0, 0]_9 + 2[2, 2, 0, 0]_9 + [2, 1, 1, 0]_9 + 7[2, 1, 0, 1]_9 + 12[2, 1, 0, 0]_9 + [2, 0, 1, 1]_9 \\ & + 7[2, 0, 1, 0]_9 + 7[2, 0, 0, 2]_9 + 19[2, 0, 0, 1]_9 + 10[2, 0, 0, 0]_9 + [1, 2, 0, 1]_9 + 5[1, 2, 0, 0]_9 + 2[1, 1, 1, 0]_9 \\ & + [1, 1, 0, 2]_9 + 14[1, 1, 0, 1]_9 + 17[1, 1, 0, 0]_9 + 4[1, 0, 1, 1]_9 + 15[1, 0, 1, 0]_9 + 9[1, 0, 0, 2]_9 + 22[1, 0, 0, 1]_9 \\ & + 12[1, 0, 0, 0]_9 + [0, 3, 0, 0]_9 + 2[0, 2, 0, 1]_9 + 5[0, 2, 0, 0]_9 + 6[0, 1, 1, 0]_9 + 3[0, 1, 0, 2]_9 + 15[0, 1, 0, 1]_9 \\ & + 13[0, 1, 0, 0]_9 + 5[0, 0, 1, 1]_9 + 10[0, 0, 1, 0]_9 + 2[0, 0, 0, 3]_9 + 10[0, 0, 0, 2]_9 + 12[0, 0, 0, 1]_9 + 3[0, 0, 0, 0]_9) q^9 \\ & + O(q^{10}). \end{aligned} \quad (3.19)$$

The coefficients are positive integers. These results agree with Appendix B of [3] where they were computed order by order in terms of $SO(8)$ representations, and then reconstituted into $SO(9)$ representations. Using our generating formula, we easily produce covariant results at higher orders, as demonstrated above.

We point out a selection rule which arises up to this order in the mass expansion which states that the third entry in the set of Dynkin labels is either 0 or 1 but not higher. It would be interesting to attempt to prove such a selection rule for the full perturbative spectrum.

Factoring symmetric tensors

We note that Z_m starts with 1, suggesting it is appropriate to take a plethystic logarithm. The result to order q^2 is

$$PL[Z_m] = [1, 0, 0, 0]_9 q + ([0, 0, 0, 1]_9 - 1) q^2 + O(q^3). \quad (3.20)$$

This result implies that at order q^n there is an n -th symmetric product of the vector representation, which is a reducible representation. Indeed, this pattern is observed in equation (3.20). In order to proceed let us take a small detour and recall a general formula for completely symmetric tensor representations of orthogonal groups. See [4] for a related discussion. The symmetrization of the vector representation is naturally given by taking the plethystic exponential and it satisfies the following identity

$$\begin{aligned} PE[[1, 0, \dots, 0]_n q] &= \frac{1}{1 - q^2} \sum_{m=0}^{\infty} [m, 0, \dots, 0]_n q^m = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} [m_1, 0, \dots, 0]_n q^{m_1+2m_2} \\ &= 1 + [1, 0, \dots, 0]_n q + ([2, 0, \dots, 0]_n + 1) q^2 + \dots \end{aligned} \quad (3.21)$$

This suggests that this function can be factorized from the expression for Z_m and may make the expansion in q easier to compute. In fact, the function has a particularly simple form as a product over all weights. We quote the result for $SO(9)$,

$$PE[[1, 0, 0, 0]_9 q] = \frac{1}{(1 - qy_1) \left(1 - \frac{qy_2}{y_1}\right) \left(1 - \frac{qy_3}{y_2}\right) \left(1 - \frac{qy_4^2}{y_3}\right) \left(1 - \frac{qy_3}{y_4}\right) \left(1 - \frac{qy_2}{y_3}\right) \left(1 - \frac{qy_1}{y_2}\right) \left(1 - \frac{q}{y_1}\right) (1 - q)}. \quad (3.22)$$

Intuitively speaking, for calculational reasons, we are factoring the contribution from nine oscillator modes at level one although not all are physical in the light-cone and we will therefore pay a price. To proceed, define

$$Z_m = PE[[1, 0, \dots, 0]_n q] Z'_m, \quad (3.23)$$

Again, this is done for ease of computation. Since not all of the factored modes are physical, there can now be negative signs in the remaining expression. For completeness we write down the expansion of Z'_m to seventh order

$$\begin{aligned} Z'_m &= 1 + ([0, 0, 0, 1]_9 - 1) q^2 + ([0, 1, 0, 0]_9 + [1, 0, 0, 0]_9 - [0, 0, 0, 1]_9) q^3 \\ &+ ([1, 0, 0, 1]_9 + [0, 0, 0, 1]_9 - [1, 0, 0, 0]_9) q^4 \\ &+ ([0, 0, 0, 2]_9 + [1, 1, 0, 0]_9 + [2, 0, 0, 0]_9 + 1) q^5 \\ &+ ([2, 0, 0, 1]_9 + [0, 1, 0, 1]_9 + [0, 0, 1, 0]_9 + [1, 0, 0, 1]_9 + [1, 0, 0, 0]_9 + 1) q^6 \\ &+ ([1, 0, 0, 2]_9 + [2, 1, 0, 0]_9 + [3, 0, 0, 0]_9 + [0, 1, 0, 1]_9 + [1, 0, 0, 1]_9 + [1, 0, 1, 0]_9 \\ &+ [1, 1, 0, 0]_9 + [0, 0, 1, 0]_9 + [0, 1, 0, 0]_9 + [0, 0, 0, 1]_9 + [1, 0, 0, 0]_9 - 1) q^7 \\ &+ O(q^8) \end{aligned} \quad (3.24)$$

This expression is an improved version of equation (3.20) in the sense that it contains less terms. One can now repeat this calculational simplification with higher oscillator modes if one so desires.

3.2 Bosonic string theory

Before we turn to applications of the above results, we show how the same method applies to bosonic string theories in twenty-six dimensions. For the left-movers one obtains the partition function:

$$Z_{Bosonic} = \frac{1}{q} PE \left[\frac{q}{1 - q} ([1, 0, \dots, 0]_{25} - 1) \right]. \quad (3.25)$$

The character of the vector representation of $SO(25)$ can be taken to be the following function of the fugacities x_1, \dots, x_{12} :

$$[1, 0, \dots, 0]_{25} = \sum_{\alpha} x^{\alpha} = x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \dots + \frac{x_{11}}{x_{10}} + \frac{x_{12}^2}{x_{11}} + \frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_{10}}{x_{11}} + \frac{x_{11}}{x_{12}^2} + 1, \quad (3.26)$$

where α runs over the set of weights of the representation and x^{α} is a multi-index notation. The first few terms in the expansion are

$$\begin{aligned} Z_{Bosonic} &= \frac{1}{q} + [1, 0, \dots, 0]_{24} + [2, 0, \dots, 0]_{25}q \\ &+ ([3, 0, \dots, 0]_{25} + [0, 1, 0, \dots, 0]_{25})q^2 \\ &+ ([4, 0, \dots, 0]_{25} + [2, 0, \dots, 0]_{25} + [1, 1, 0, \dots, 0]_{25} + 1)q^3 \\ &+ ([5, 0, \dots, 0]_{25} + [3, 0, \dots, 0]_{25} + [2, 1, 0, \dots, 0]_{25} \\ &+ [1, 1, 0, \dots, 0]_{25} + [1, 0, \dots, 0]_{25} + [0, 1, 0, \dots, 0]_{25})q^4 + O(q^5). \end{aligned} \quad (3.27)$$

We can easily combine chiral halves and level match to obtain the closed bosonic string spectrum in twenty-six dimensions.

3.3 Superstring Theories

To write down the partition function for the superstring theories, we need to expand the chiral partition function (3.16) in powers of q ,

$$Z_{Left}(q; z_1, z_2, z_3, z_4) = \sum_{n=0}^{\infty} d_n(z_1, z_2, z_3, z_4) q^n, \quad (3.28)$$

where d_n for $n > 1$ is a sum of fermionic and bosonic representations of $SO(9)$. The first few terms can be read from equation (3.20)

$$\begin{aligned} d_0 &= [1, 0, 0, 0]_8 + [0, 0, 0, 1]_8 \\ d_1 &= [2, 0, 0, 0]_9 + [1, 0, 0, 1]_9 + [0, 0, 1, 0]_9 \\ d_2 &= [1, 0, 0, 0]_9 d_1 \end{aligned} \quad (3.29)$$

etcetera.

3.3.1 Type II

It should be clear that the generating function of Type II string theories can be written as the product of $SO(9)$ covariant left-moving and right-moving partition functions. To obtain the physical spectrum it is sufficient to implement level matching. The massive spectrum at level n is given by the tensor product of d_n 's.

$$\sum_{n=1}^{\infty} d_n(z_1, z_2, z_3, z_4) \tilde{d}_n(z_1, z_2, z_3, z_4) q^n \bar{q}^n. \quad (3.30)$$

For Type IIB string theory, we have $\tilde{d}_n = d_n$ while for Type IIA we switch the chirality of the zero-modes on the right: $\tilde{d}_0 = [1, 0, 0, 0]_8 + [0, 0, 1, 0]_8$ and $\tilde{d}_n = d_n$ for $n \geq 1$.

3.3.2 Type I

At one loop in Type I string theory, we have contributions from the torus, the Klein bottle, the annulus and the Möbius strip. The contributions from the torus and Klein bottle take the form:

$$Z_{T+KB}(q; z_1, z_2, z_3, z_4) = \sum_{n=0}^{\infty} [d_n(z_1, z_2, z_3, z_4)]_{S,A}^2 q^n \bar{q}^n, \quad (3.31)$$

where the subscript S, A mean that we symmetrize bosonic representations in the chiral partition function Z_{Left} while we anti-symmetrize fermions in Z_{Left} . The net effect for space-time bosons is to pick up the symmetric part in the NS-NS sector and the anti-symmetric part of the R-R sector of Type IIB string theory. Note that we already level-matched the spectrum. The contribution from the annulus and the Möbius strip take the form

$$\begin{aligned} Z_{A+M}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) &= [0, 1, 0, \dots, 0]_{32} \sum_{n=0}^{\infty} d_{2n}(z_1, z_2, z_3, z_4) q^{2n} \\ &+ ([2, 0, \dots, 0]_{32} + 1) \sum_{n=0}^{\infty} d_{2n+1}(z_1, z_2, z_3, z_4) q^{2n+1}. \end{aligned} \quad (3.32)$$

We have taken the opportunity to also introduce fugacities that indicate the $SO(32)$ representation content of the Chan-Paton degrees of freedom associated to the open strings. The Type I partition function is the sum

$$Z_{TypeI}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) = Z_{T+KB} + Z_{A+M}. \quad (3.33)$$

Combining these equations, we find the familiar massless sector:

$$\begin{aligned} Z_{0, TypeI} &= [2, 0, 0, 0]_8 + [1, 0, 0, 1]_8 + [0, 1, 0, 0]_8 + [0, 0, 1, 0]_8 + [0, 0, 0, 0]_8 \\ &+ [0, 1, 0, \dots, 0]_{32}([1, 0, 0, 0]_8 + [0, 0, 0, 1]_8). \end{aligned} \quad (3.34)$$

We define the massive partition function:

$$Z_{m,I}(q; y_1, y_2, y_3, y_4) = \frac{Z_{TypeI} - Z_{0, TypeI}}{Z_Q}, \quad (3.35)$$

where Z_Q is the supermultiplet given in equation (3.4). The massive partition function $Z_{m,I}$ has the expansion:

$$\begin{aligned} Z_{m,I} &= q([2, 0, \dots, 0]_{32} + 1 + [0, 1, 0, \dots, 0]_{32}[1, 0, 0, 0]_9 q + \dots) \\ &+ q\bar{q}([2, 0, 0, 0]_9 + [0, 0, 1, 0]_9 \\ &+ ([4, 0, 0, 0]_9 + [2, 1, 0, 0]_9 + [2, 0, 1, 0]_9 + [2, 0, 0, 1]_9 + 2[2, 0, 0, 0]_9 + [1, 1, 0, 1]_9 \\ &+ [1, 1, 0, 0]_9 + [1, 0, 0, 2]_9 + 2[1, 0, 0, 1]_9 + [0, 2, 0, 0]_9 + [0, 1, 0, 1]_9 + [0, 1, 0, 0]_9 \\ &+ [0, 0, 1, 1]_9 + 2[0, 0, 1, 0]_9 + [0, 0, 0, 1]_9 + [0, 0, 0, 0]_9)q\bar{q} + \dots). \end{aligned} \quad (3.36)$$

The power of q in the open string contributions is equal to the mass squared in string units, $\alpha' m^2$, while the power of $q\bar{q}$ in the closed string contributions is $\alpha' m^2/2$.

3.3.3 Heterotic $SO(32)$

Let's discuss the $SO(32)$ heterotic string theory next. Again, we have a $SO(32)$ gauge group, but including the fugacities that code the gauge group representation content of all excitations is now slightly more involved. The charged sector is generated by thirty-two fermionic generators and hence we define the fermionic plethystic exponential

$$Z_{F,32}(q; f; s_1, \dots, s_{16}) = PEF \left[\frac{f}{1-q} [1, 0, \dots, 0]_{32} \right], \quad (3.37)$$

where s_1, \dots, s_{16} are the 16 fugacities of $SO(32)$, and below only the f dependence is explicit, namely $Z_{F,32}(f)$. The NS contribution is as above,

$$Z_{NS,32} = \frac{1}{2q} (Z_{F,32}(\sqrt{q}) + Z_{F,32}(-\sqrt{q})), \quad (3.38)$$

and similarly the R sector gets 2 contributions from

$$Z_{32,\pm} = \frac{q}{2} (Z_{F,32}(q) \pm Z_{F,32}(-q)). \quad (3.39)$$

The RNS contribution takes the form

$$Z_{RNS,32}(q; s_1, \dots, s_{16}) = Z_{NS,32} + [0, \dots, 0, 1]_{32} Z_{32,+} + [0, \dots, 0, 1, 0]_{32} Z_{32,-}. \quad (3.40)$$

We further need the contribution from the eight bosonic oscillators, as in equation (3.8), to construct the right moving sector

$$Z_{Right,32}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) = Z_B Z_{RNS,32}. \quad (3.41)$$

The first terms read

$$\begin{aligned} Z_{Right,32}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) = & \frac{1}{q} + [1, 0, 0, 0]_8 + [0, 1, 0, \dots, 0]_{32} \\ & + ([2, 0, 0, 0]_9 + [1, 0, 0, 0]_9 [0, 1, 0, \dots, 0]_{32} + [2, 0, \dots, 0]_{32} \\ & + [0, 0, 0, 1, 0, \dots, 0]_{32} + [0, \dots, 0, 1]_{32} + 1) q \\ & + ([3, 0, 0, 0]_9 + [2, 0, 0, 0]_9 [0, 1, 0, \dots, 0]_{32} + [0, 1, 0, 0]_9 \\ & + [1, 0, 0, 0]_9 ([2, 0, \dots, 0]_{32} + [0, 1, 0, \dots, 0]_{32} + [0, 0, 0, 1, 0, \dots, 0]_{32} + [0, \dots, 0, 1]_{32} + 1) \\ & + [1, 0, 1, 0, \dots, 0]_{32} + [1, 0, \dots, 0, 1, 0]_{32} + 2[0, 1, 0, \dots, 0]_{32} + [0, 0, 0, 0, 0, 1, 0, \dots, 0]_{32}) q^2 \\ & + ([4, 0, 0, 0]_9 + [3, 0, 0, 0]_9 [0, 1, 0, \dots, 0]_{32} + [1, 1, 0, 0]_9 \\ & + [2, 0, 0, 0]_9 ([2, 0, \dots, 0]_{32} + [0, 1, 0, \dots, 0]_{32} + [0, 0, 0, 1, 0, \dots, 0]_{32} + [0, \dots, 0, 1]_{32} + 2) \\ & + [1, 0, 0, 0]_9 ([2, 0, \dots, 0]_{32} + [1, 0, 1, 0, \dots, 0]_{32} + [1, 0, \dots, 0, 1, 0]_{32} + 3[0, 1, 0, \dots, 0]_{32} \\ & + [0, 0, 0, 1, 0, \dots, 0]_{32} + [0, 0, 0, 0, 0, 1, 0, \dots, 0]_{32} + [0, \dots, 0, 1]_{32} + 1) \\ & + [0, 1, 0, 0]_9 [0, 1, 0, \dots, 0]_{32} + 2[2, 0, \dots, 0]_{32} + [1, 0, 1, 0, \dots, 0]_{32} + [1, 0, 0, 0, 1, 0, \dots, 0]_{32} \\ & + [1, 0, \dots, 0, 1, 0]_{32} + [0, 2, 0, \dots, 0]_{32} + [0, 1, 0, \dots, 0, 1]_{32} \\ & + [0, 1, 0, \dots, 0]_{32} + 2[0, 0, 0, 1, 0, \dots, 0]_{32} \\ & + [0, 0, 0, 0, 0, 0, 0, 1, 0, \dots, 0]_{32} + [0, \dots, 0, 1]_{32} + 3) q^3 + O(q^4). \end{aligned} \quad (3.42)$$

This expression satisfies the well known condition which states that only two out of the four conjugacy classes of $SO(32)$ are present in the perturbative spectrum - the adjoint class and the spinor class. In terms of the sixteen Dynkin labels this condition means that the sum of odd entries should be 0 mod 2. Indeed one can check that all representations of $SO(32)$ in (3.42) satisfy this condition.

To get the final answer, one combines the right-moving partition function with the left-moving supersymmetric partition function and level-matches.

3.3.4 Heterotic $E_8 \times E_8$

For the $E_8 \times E_8$ theory, we proceed similarly. Define the fermionic plethystic exponential for $SO(16)$

$$Z_{F,16}(q; f; s_1, \dots, s_8) = P_{EF} \left[\frac{f}{1-q} [1, 0, \dots, 0]_{16} \right], \quad (3.43)$$

where s_1, \dots, s_8 are the 8 fugacities of $SO(16)$, and below only the f dependence is explicit, $Z_{F,16}(f)$. The NS contribution is

$$Z_{NS,16} = \frac{Z_{F,16}(\sqrt{q}) + Z_{F,16}(-\sqrt{q})}{2\sqrt{q}}, \quad (3.44)$$

and the R sector gets two contributions from

$$Z_{16,\pm} = \frac{\sqrt{q}}{2} (Z_{F,16}(q) \pm Z_{F,16}(-q)). \quad (3.45)$$

The RNS contribution takes the form

$$Z_{RNS,16}(q; s_1, \dots, s_8) = Z_{NS,16} + [0, \dots, 0, 1]_{16} Z_{16,+} + [0, \dots, 0, 1, 0]_{16} Z_{16,-}. \quad (3.46)$$

Collecting all contributions, including the 8 bosonic oscillators, taken from equation (3.8) the right moving sector becomes

$$Z_{Right,E8 \times E8}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) = Z_B Z_{RNS,16}(q, s_1, \dots, s_8) Z_{RNS,16}(q, s_9, \dots, s_{16}).$$

The first few terms read

$$\begin{aligned}
Z_{Right, E8 \times E8}(q; z_1, z_2, z_3, z_4; s_1, \dots, s_{16}) &= \frac{1}{q} \\
&+ [1, 0, 0, 0]_8 + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2} \\
&+ ([2, 0, 0, 0]_9 + [1, 0, 0, 0]_9 ([0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2})) \\
&+ [1, 0, 0, 0, 0, 0, 0, 0]_{E8_1} + [1, 0, 0, 0, 0, 0, 0, 0]_{E8_2} \\
&+ [0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2} + 2)q \\
&+ ([3, 0, 0, 0]_9 + [0, 1, 0, 0]_9 + 2[1, 0, 0, 0]_9) \\
&+ [2, 0, 0, 0]_9 ([0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2}) \\
&+ [1, 0, 0, 0]_9 ([1, \dots, 0]_{E8_1} + [1, \dots, 0]_{E8_2} + [0, \dots, 0, 1]_{E8_1} + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2}) \\
&+ [0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2} + [1, 0, 0, 0, 0, 0, 0, 0]_{E8_1} [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2} \\
&+ [0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} [1, 0, 0, 0, 0, 0, 0, 0]_{E8_2} + [0, 0, 0, 0, 0, 0, 0, 1]_{E8_1} [0, 0, 0, 0, 0, 0, 0, 1]_{E8_2} \\
&+ [0, 0, 0, 0, 0, 0, 1, 0]_{E8_1} + [0, 0, 0, 0, 0, 0, 1, 0]_{E8_2} + 2[0, \dots, 0, 1]_{E8_1} + 2[0, \dots, 0, 1]_{E8_2})q^2 \\
&+ O(q^3).
\end{aligned} \tag{3.47}$$

For reference we record the dimensions of the $E8$ representations which enter into this formula,

$$\begin{aligned}
dim[0, 0, 0, 0, 0, 0, 0, 1]_{E8} &= 248, \\
dim[1, 0, 0, 0, 0, 0, 0, 0]_{E8} &= 3875, \\
dim[0, 0, 0, 0, 0, 0, 1, 0]_{E8} &= 30380.
\end{aligned} \tag{3.48}$$

It is possible to treat the $E_8 \times E_8$ quantum numbers covariantly from the start by bosonizing the fermions, and using that the momentum lattice of the bosons is the $E_8 \times E_8$ root lattice.

4 Conclusions and some open problems

We gave the $SO(D-1)$ covariant form of massive string spectra in $\mathbb{R}^{D-1,1}$ for various string theories. Our method generalizes to other Minkowski compactifications of string theory. We already illustrated how one can generate further generalizations of partition functions that also code the gauge group representation content at all mass levels. Our method is also applicable to compactifications with isometries, or super-isometries where we may choose to introduce fugacities for the Cartan subalgebra of the super-isometry group. Generating functions that code the representation content of compactifications with discrete symmetries (like Calabi-Yau compactifications at Gepner points) can also be written down. These are fairly straightforward generalizations of our results, and the resulting counting functions are likely to have interesting (modular and other) properties.

It is a further challenge to generalize these counting functions to other symmetric string backgrounds, including AdS space-times.

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